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D-branes on group manifolds and fusion rings*

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Abstract: In this paper we compute the charge group for symmetry preserving D-branes on group manifolds for all simple, simply-connected, connected compact Lie groups G.

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1. Introduction

D-branes on group manifolds have been the subject of intensive research over the past few years (see, e.g., [1]–[23]), as they provide a good laboratory for testing our intuitive understanding of D-branes on curved spaces and a case where both geometric and algebraic methods can be applied.

Algebraically, D-branes on a group manifold G are obtained as solutions to the gluing conditions of the chiral currents of the WZW-model [3, 6, 7, 14]

$$J(z) = \omega \cdot \bar{J}(\bar{z})\,, \qquad \text{for all } z = \bar{z}\,, \tag{1.1}$$

where ω is an automorphism of the Lie algebra $\mathfrak{g}$ of G. In this paper we will restrict ourselves to the sector $\omega = 1$, i.e. the so-called symmetry preserving (or untwisted) D-branes.

Geometrically, symmetry preserving D-branes wrap conjugacy classes $\mathcal{C}(h) = \{ghg^{-1} : g \in G\}$. Even though the conjugacy classes $\mathcal{C}(h)$ are contractible within G, the D-branes correspond to a stable configuration due to the process of flux stabilization [8, 16, 17]. Upon quantization, it is known that the symmetry preserving branes are in 1–1 correspondence with the integrable highest weight modules of the affine Lie algebra $\widehat{\mathfrak{g}}$ (at level k) (see, e.g. [24]) through the so-called Ishibashi-Cardy boundary states. In the geometrical picture, the D-brane associated to a boundary state labelled by an integrable weight $\mu \in P_+^{(k)}$ wraps a conjugacy class $\mathcal{C}(h_\mu)$ where [3, 4, 6, 7]

$$h_\mu = e^{\zeta_\mu} , \qquad \zeta_\mu = (2\pi i) \frac{\mu + \rho}{k + \mathrm{h}^\vee} . \tag{1.2}$$

[More precisely, the conjugacy classes around which the D-branes are wrapped get slightly smeared out (see [5, 6]).]

In this paper we consider the charge group $C(G, k)$ for symmetry preserving D-branes on general compact, connected, simply-connected, simple Lie groups G.

By considering the process of brane condensation, it was argued in [14] that the charge associated to a D-brane labelled by $\lambda \in P_+^{(k)}$ is given by the dimension d_λ of the finite dimensional irreducible representation $L(\lambda)$ of the horizontal Lie algebra $\mathfrak{g}$, modulo some integer x, i.e. the charge group is of the form $\mathbb{Z}/x\mathbb{Z}$, where x is determined such that the d_λ satisfy the fusion rule algebra modulo x. We find the result that the charge group $C(G, k)$, for sufficiently large k, is given by $\mathbb{Z}/x\mathbb{Z}$, where

$$x = \frac{k + \mathrm{h}^\vee}{\gcd \left\{ k + \mathrm{h}^\vee, y \right\}} , \tag{1.3}$$

and y is given in table 1. This generalizes the results for $G = \mathrm{SU}(N)$ found by, in particular, [14, 16] (see also [3, 19] for $N = 2, 3$), in which case $y = \mathrm{lcm}(1, 2, \ldots, N-1)$ and $\mathrm{h}^\vee = N$. Geometrically, it has been conjectured that the charge group, in the large volume limit, should be given by the twisted K-theory of G [25]–[27]. For $\mathrm{SU}(N)$ it has been shown that the algebraic and geometric definition of the charge group are consistent. A comparison for other groups awaits the computation of the relevant twisted K-theory.

The paper is organized as follows. In section 2 we review fusion rings, in particular for the WZW conformal field theory. In section 3 we present our results. Section 4 contains some details of the proofs for the A_n and C_n cases. In section 5 we discuss some exceptions to the discussion in section 3, and in section 6 we discuss symmetry properties of the resulting D-brane charges. We conclude with a discussion in section 7. In the three appendices we summarize our conventions, provide a list of generators for the fusion ideals and discuss an alternative derivation of our results using fusion potentials.

2. Fusion rings

Consider the chiral algebra $\mathcal{A}$ of a (rational) two dimensional conformal field theory together with a (finite) set of irreducible representations, labelled by $i \in I$ (where I is some index set), closed under modular transformations of their characters. To these data one can associate a commutative, associative, unital ring $\mathcal{F}$, the so-called 'fusion ring' of $\mathcal{A}$. Explicitly, in terms of a (preferred) basis $\{\phi_i\}_{i \in I}$ of $\mathcal{F}$ we have

$$\phi_i \times \phi_j = \sum_{k \in I} N_{ij}{}^k \, \phi_k , \tag{2.1}$$

where the coefficients ${N_{ij}}^k \in \mathbb{Z}_{\geqslant 0}$ are called the fusion coefficients. The matrices N_i, with components ${(N_i)_j}^k = {N_{ij}}^k$, are mutually commuting and are simultaneously diagonalized by the modular matrix S. This leads to the Verlinde formula [28]

$$ {N_{ij}}^k = \sum_{l \in I} \frac{S_{il} S_{jl} S^*_{kl}}{S_{1l}} \,, \tag{2.2} $$

where S^*_{ij} denotes complex conjugation, and we have denoted the unit of $\mathcal{F}$ by 1. In terms of S, the eigenvalues $\lambda_i^{(l)}$ of N_i are given by

$$ \lambda_i^{(l)} = \frac{S^*_{li}}{S_{l1}} \,. \tag{2.3} $$

On abstract grounds (cf. [29]) it follows that $\mathcal{F}$ is isomorphic to a free polynomial ring factored by an ideal $\mathcal{I}$, i.e.

$$ \mathcal{F} \cong \frac{\mathbb{C}[x_1, \ldots, x_N]}{\mathcal{I}} \,. \tag{2.4} $$

In fact, when the generators $x_1, \ldots, x_N$ correspond to (a subset of) the primary fields $\phi_{i_1}, \ldots, \phi_{i_N}$, the ideal $\mathcal{I}$ can be explicitly characterized as those polynomials $P \in \mathbb{C}[x_1, \ldots, x_N]$ which vanish on the set

$$ \left\{ \left(\frac{S^*_{li_1}}{S_{l1}}, \frac{S^*_{li_2}}{S_{l1}}, \ldots, \frac{S^*_{li_N}}{S_{l1}} \right) \in \mathbb{C}^N : l \in I \right\} . $$

We refer to $\mathcal{I}$ as the 'fusion ideal' of $\mathcal{F}$. The ideals arising from the fusion rules of a RCFT are actually of a special kind, they correspond to so-called jacobian varieties (see appendix C).

In the case of the conformal field theory corresponding to the WZW model for a compact, simple, connected, simply-connected, Lie group G, the relevant representations are the integrable highest weight modules of the associated (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$ at level k. They are parametrized by weights $\widehat{\lambda} = (\lambda, k) \in P_+^{(k)}$ (see appendix A for a list of our Lie algebra conventions and notations).

The modular S-matrix can be expressed as

$$ \frac{S^*_{\mu\lambda}}{S_{\mu 1}} = \chi_\lambda(\zeta_\mu) \,, \qquad \zeta_\mu = (2\pi i) \frac{\mu + \rho}{k + \mathrm{h}^\vee} \,, \tag{2.5} $$

where χ_λ is the character of the finite dimensional irreducible representation $L(\lambda)$ with dominant integral weight λ, defined as

$$ \chi_\lambda(\sigma) = \frac{\sum_{w \in W} \epsilon(w) e^{(w(\lambda+\rho)|\sigma)}}{\sum_{w \in W} \epsilon(w) e^{(w(\rho)|\sigma)}} \,, \tag{2.6} $$

for $\sigma \in \mathfrak{h}^*$. [We identify $\mathfrak{h}^*$ with $\mathfrak{h}$, by means of the nondegenerate bilinear form $(\ |\ )$, throughout.]

It follows that

$$ \chi_\lambda(\zeta_\alpha) \chi_\mu(\zeta_\alpha) = \sum_{\nu \in P_+^{(k)}} {N_{\lambda\mu}}^\nu \, \chi_\nu(\zeta_\alpha) \,, \tag{2.7} $$

for all $\alpha \in P_+^{(k)}$. That is, we can think of the fusion rules of an affine Lie algebra CFT as truncated tensor products. [In fact, this leads to a useful algorithm for determining explicit fusion rules, the so-called Kac-Walton algorithm, which is a generalization of the Racah-Speiser algorithm for determining tensor product coefficients (see, e.g., [30] and references therein).]

Thus we have the following characterization of the fusion ring for a WZW model at level k

$$\mathcal{F}_k = \frac{\mathbb{Z}[\chi_1, \ldots, \chi_n]}{\mathcal{I}_k}\,, \tag{2.8}$$

where χ_i denotes the character of the fundamental representation with highest weight Λ_i, and where the fusion ideal can be explicitly described as those elements in the Grothendieck ring of characters which vanish on the all the points ζ_μ, $\mu \in P_+^{(k)}$.

Elements in $\mathcal{I}_k$ can easily be constructed by making use of the identity (see, e.g., [30])

$$\chi_{w\cdot\lambda}(\zeta_\mu) = \epsilon(w)\chi_\lambda(\zeta_\mu)\,, \tag{2.9}$$

for all $w \in W_{\mathrm{aff}}$, where $w \cdot \lambda = w(\lambda + \rho) - \rho$ denotes the shifted action of the affine Weyl group (see appendix A for more details). This implies that $\chi_{w\cdot\lambda} - \epsilon(w)\chi_\lambda \in \mathcal{I}_k$ for all $\lambda \in P_+$ and $w \in W_{\mathrm{aff}}$. In particular, we conclude that $\chi_\lambda \in \mathcal{I}_k$ if there exists a positive root $\alpha \in \Delta_+$ such that $(\lambda + \rho|\alpha) \equiv 0 \mod (k + \mathrm{h}^\vee)$, as this implies $r_\alpha \cdot \lambda = \lambda$, where r_α denotes the reflection in the root α, and hence $\chi_\lambda(\zeta_\mu) = 0$, for all $\mu \in P_+^{(k)}$. For example, we have $\chi_\lambda \in \mathcal{I}_k$ for all weights λ on the hyperplane $(\lambda|\theta) = k + 1$. Let us call such a weight a 'boundary weight' (at level k). Thus all the characters corresponding to a level-k boundary weight are in the fusion ideal $\mathcal{I}_k$. In fact, we believe that the fusion ideal $\mathcal{I}_k$ is generated by the set of characters corresponding to the boundary weights (this is true for A_n [29, 31]).[1]

While this would characterize the fusion ideal in terms of a finite set of generators, the number of boundary weights grows rapidly with k and the corresponding set of generators is still far from optimal. In appendix B we provide what we believe is a substantially smaller set of boundary weights, whose characters generate $\mathcal{I}_k$. For A_n and C_n this has been proven. The sets provided are not always optimal, but it appears that what is an optimal set in those cases behaves chaotically with respect to k.

3. D-brane charge group

The charge group for symmetry preserving D-branes on group manifolds was investigated in [11]–[14], [16]. In particular, by considering the process of D-brane creation and annihilation, it was argued in [14] that the charge of a D-brane associated to an integrable weight $\lambda \in P_+^{(k)}$ is given by the dimension d_λ of the finite dimensional irreducible representation $L(\lambda)$ of the horizontal Lie algebra $\mathfrak{g}$, modulo some integer x. That is, the charge group is of the form $\mathbb{Z}/x\mathbb{Z}$, where x is determined (as the maximal integer) such that the d_λ satisfy

[1] With the exception of E_8 level 2, see appendix B.

the fusion rule algebra[2] modulo x

$$d_\lambda d_\mu = \sum_{\nu \in P_+^{(k)}} N_{\lambda\mu}{}^{\nu}\, d_\nu \mod x\,. \tag{3.1}$$

Of course, the dimensions d_λ give rise to a homomorphism

$$\dim : \mathbb{Z}[\chi_1, \ldots, \chi_n] \to \mathbb{Z}\,, \qquad \chi_\lambda \mapsto d_\lambda\,, \tag{3.2}$$

so the problem can be rephrased as the determination of the maximal value of $x \in \mathbb{N}$, such that the dimension homomorphism (3.2) descends to

$$\dim' : \mathcal{F}_k = \frac{\mathbb{Z}[\chi_1, \ldots, \chi_n]}{\mathcal{I}_k} \to \frac{\mathbb{Z}}{x\mathbb{Z}}\,. \tag{3.3}$$

Clearly, a necessary and sufficient condition for dim to descend to a map on $\mathcal{F}_k$ is that $\dim(\mathcal{I}_k) \subset x\mathbb{Z}$. Note, however, that this analysis presumes a particular presentation of $\mathcal{F}_k$ as the quotient of the whole character ring $\mathbb{Z}[\chi_1, \ldots, \chi_n]$. When not all the fundamental representations are integrable at the level k considered, i.e. when $k < \max\{a_i^\vee\}$, the fusion ring $\mathcal{F}_k$ can be presented as a quotient of a subset of the character ring, and the condition considered above is too strong for a homomorphism $\mathcal{F}_k \to \mathbb{Z}/x\mathbb{Z}$ to exist. We will discuss this further in section 5, but for now we analyze the condition $\dim(\mathcal{I}_k) \subset x\mathbb{Z}$.

It is sufficient to check $\dim(\mathcal{I}_k) \subset x\mathbb{Z}$ on a set of generators of $\mathcal{I}_k$. In particular, assuming the set of boundary weights generate the fusion ideal, as discussed in section 2 we have,

$$x = \gcd\left\{d_\lambda : (\lambda|\theta) = k+1\right\}\,. \tag{3.4}$$

The main result of our paper is that the maximal value of x for which $\dim(\mathcal{I}_k) \subset x\mathbb{Z}$ is given by the following formula

$$x = \frac{k + \mathrm{h}^\vee}{\gcd\left\{k + \mathrm{h}^\vee, y\right\}}\,, \tag{3.5}$$

where y is an integer, independent of k, given in table 1.

This generalizes a result in [14] (see also [16]) for $\mathfrak{g} = A_n$. Note that in all cases we can write $y = \mathrm{lcm}\{y_\alpha\}$, for some integers y_α. For the simply laced Lie algebras, these integers are found to be consecutive, from 1 up to $\mathrm{h}^\vee - 1$, but for non-simply laced Lie algebras, additional integers are required. Note, moreover, that for all Lie algebras, with the exception of C_n, y is given by the lowest common multiple of all integers from 1 up to $\mathrm{h} - 1$ (cf. the discussion in section 6).

We will prove eq. (3.5) in section 4 for the algebras A_n and C_n. In the other cases we have strong numerical evidence, based on explicit computations for levels up to $k = 5000$, using both the explicit fusion rules (calculated with the help of [32]) and sets of elements in the ideal constructed with the procedure outlined in section 2. In appendix B we provide what we believe is a set of generators for the ideal $\mathcal{I}_k$.

[2] In fact, this is a necessary condition. It is not clear whether this condition is sufficient in all cases. Cf. section 6.

$\mathfrak{g}$	h	$\mathrm{h}^\vee$	y
A_n	$n+1$	$n+1$	$\mathrm{lcm}\,\{1,2,\ldots,n\}$
B_n	$2n$	$2n-1$	$\mathrm{lcm}\,\{1,2,\ldots,2n-1\}$
C_n	$2n$	$n+1$	$\mathrm{lcm}\,\{1,2,\ldots,n,1,3,5,\ldots,2n-1\}$
D_n	$2n-2$	$2n-2$	$\mathrm{lcm}\,\{1,2,\ldots,2n-3\}$
E_6	12	12	$\mathrm{lcm}\,\{1,2,\ldots,11\}$
E_7	18	18	$\mathrm{lcm}\,\{1,2,\ldots,17\}$
E_8	30	30	$\mathrm{lcm}\,\{1,2,\ldots,29\}$
F_4	12	9	$\mathrm{lcm}\,\{1,2,\ldots,11\}$
G_2	6	4	$\mathrm{lcm}\,\{1,2,\ldots,5\}$

Table 1: The Coxeter number h and dual Coxeter number $\mathrm{h}^\vee$ for each (finite dimensional) simple Lie algebra, and the number y occurring in eq. (3.5).

For an intuitive way to see how the formula for x may arise, let us consider the Weyl dimension formula

$$d_\lambda = \prod_{\alpha\in\Delta_+} \frac{(\lambda+\rho|\alpha)}{(\rho|\alpha)}\,. \tag{3.6}$$

When λ is a boundary weight, we find that $(\lambda+\rho|\theta)$ contributes a factor $k+\mathrm{h}^\vee$ to the numerator, so it is not surprising that it appears in the formula for the greatest common divisor of these dimensions.

Let us now concentrate on the factors $(\rho|\alpha)$ of the denominator. For simply laced Lie algebras these factors run from 1 up to $(\rho|\theta)=\mathrm{h}^\vee-1$ (with some repetitions). These are the y_α whose least common multiple gives the parameter y in (3.5). For non-simply laced algebras, the factors $(\rho|\alpha)$ need not be integers. It is easily verified that setting

$$y_\alpha = \begin{cases} (\rho|\alpha) & \text{if } (\rho|\alpha)\in\mathbb{Z}\,,\\ (\rho|\alpha^\vee) & \text{if } (\rho|\alpha)\notin\mathbb{Z}\,,\end{cases} \tag{3.7}$$

and taking $y=\mathrm{lcm}\,\{y_\alpha : \alpha\in\Delta_+\}$, reproduces the results for y given in table 1.

4. Analytical Results

In this section we provide explicit proofs of (3.5) for the algebras A_n and C_n. While the result for A_n has already been established in [14, 16], using simple currents and outer automorphisms, respectively, we will present an alternative proof which generalizes, at least in principle, to the other simple Lie algebras.

4.1 A_n

We claim (see (3.5)) that for symmetry preserving D-branes defined on the group A_n, ($n\geqslant 1$), at levels $k\geqslant 1$, the charge group has the form $\mathbb{Z}/x\mathbb{Z}$, where

$$x = \frac{k+n+1}{\gcd\,\{k+n+1, \mathrm{lcm}\,\{1,2,\ldots,n\}\}}\,. \tag{4.1}$$

The first proof we present is based on the results in [14]. There it is shown by an induction argument, using the simple currents of A_n, that the fusion ideal is generated by the characters of the boundary modules with weights $\{k\Lambda_1 + \Lambda_i : i = 1, \ldots, n\}$. We thus have

$$x = \gcd\left\{d_{k\Lambda_1+\Lambda_i} : i = 1, 2, \ldots, n\right\} . \tag{4.2}$$

To prove that this equals (4.1) we proceed as follows: The dimensions $d_{k\Lambda_1+\Lambda_i}$ are given explicitly by

$$d_{k\Lambda_1+\Lambda_i} = \frac{i}{k+i}\binom{n+1}{i}\binom{k+n+1}{k} = \binom{k+i-1}{k}\binom{k+n+1}{n+1-i} . \tag{4.3}$$

The second expression immediately shows that

$$\gcd\left\{d_{k\Lambda_1+\Lambda_i} : i = 1, \ldots, n\right\} \geqslant \gcd\left\{\binom{k+n+1}{i} : i = 1, \ldots, n\right\} . \tag{4.4}$$

To show the reverse inequality, we prove the following relation

$$\sum_{i=1}^{n+1-j} (-1)^{i-1}\binom{n+1-i}{j} d_{k\Lambda_1+\Lambda_i} = \binom{k+n+1}{j} . \tag{4.5}$$

Substituting the first expression in (4.3) for $d_{k\Lambda_1+\Lambda_i}$ into (4.5), we find that this relation reduces to

$$\binom{k+M}{k}\sum_{i=0}^{M}(-1)^{i-1}\frac{i}{k+i}\binom{M}{i} = 1 ,$$

where we have put $M = n+1-j$. Using $\frac{i}{k+i} = 1 - \frac{k}{k+i}$ and $\sum_{i=0}^{M}(-1)^i\binom{M}{i} = 0$, we reduce this further, obtaining

$$k\binom{k+M}{k}\sum_{i=0}^{M}(-1)^{i}\frac{1}{k+i}\binom{M}{i} = 1 .$$

This equation is clearly true for $M = 0$ and all $k \in \mathbb{Z}_+$, so we can finish the proof of the relation by induction on M (for all k). In the induction step we use

$$\binom{M+1}{i} = \binom{M}{i} + \binom{M}{i-1} , \tag{4.6}$$

and change the summation variable in the sum arising from the second term from i to $i+1$. This then establishes eq. (4.5) and, hence, the equality sign in eq. (4.4). Finally, it is proven in [16, appendix C] that

$$\gcd\left\{\binom{k+n+1}{i} : i = 1, \ldots, n\right\} = \frac{k+n+1}{\gcd\left\{k+n+1, \mathrm{lcm}\left\{1, 2, \ldots, n\right\}\right\}} , \tag{4.7}$$

completing the proof of eq. (4.1).

It is clear from the discussion in section 2 that the charge group parameter x may be obtained from any set of generators of the fusion ideal. It is obviously highly desirable to find generators for which the above evaluation gives nice expressions. The proof given above uses the generators corresponding to the weights $\{k\Lambda_1 + \Lambda_i : i = 1, \dots, n\}$ whose dimensions are a little cumbersome to manipulate — hence the proof is not straightforward. An alternative set of generators for the fusion ideal was constructed in [29] (see also [33]), and given by

$$\{(k+i)\,\Lambda_1 : i = 1, \dots, n\} \,, \tag{4.8}$$

This set is particularly nice because

$$d_{(k+i)\Lambda_1} = \binom{k+i+n}{n} . \tag{4.9}$$

We now have

$$\begin{aligned} x &= \gcd\left\{\binom{k+n+1}{n}, \binom{k+n+2}{n}, \dots, \binom{k+2n}{n}\right\} \\ &= \gcd\left\{\binom{k+n+1}{1}, \binom{k+n+2}{2}, \dots, \binom{k+2n}{n}\right\} \\ &= \gcd\left\{\binom{k+n+1}{1}, \binom{k+n+1}{2}, \dots, \binom{k+n+1}{n}\right\} , \end{aligned} \tag{4.10}$$

by repeatedly using relation (4.6). With eq. (4.7), this completes the proof of (4.1). Another reason for why the set of generators (4.8) is nice is that this set is intimately related to the fusion potential for A_n [29] (see appendix C).

4.2 C_n

For C_n ($n \geqslant 2$), at levels $k \geqslant 1$, the result (3.5) is explicitly given by

$$x = \frac{k+n+1}{\gcd\{k+n+1, \operatorname{lcm}\{1,2,\dots,n,1,3,5,\dots,2n-1\}\}} . \tag{4.11}$$

Consider, as in the second A_n proof, the set of weights $\{(k+i)\,\Lambda_1 : i = 1, \dots, n\}$. While, contrary to the A_n case, these weights in general do not correspond to elements of the fusion ideal, we can use them to construct elements in the ideal by using the (shifted action of the) affine Weyl reflection r_0, as explained in section 2. We find that

$$r_0 \cdot (k+i)\,\Lambda_1 = (k+2-i)\,\Lambda_1 . \tag{4.12}$$

Hence, the following set of combinations of characters are in the fusion ideal $\mathcal{I}_k$[3]

$$\{\chi_{(k+1)\Lambda_1}, \chi_{(k+2)\Lambda_1} + \chi_{k\Lambda_1}, \chi_{(k+3)\Lambda_1} + \chi_{(k-1)\Lambda_1}, \dots, \chi_{(k+n)\Lambda_1} + \chi_{(k+2-n)\Lambda_1}\} . \tag{4.13}$$

In fact, as shown in [33, 34, 35], these combinations actually constitute a set of generators of $\mathcal{I}_k$.

[3] At low levels, some of these weights will lie outside the fundamental Weyl chamber, and have to be reflected back to the fundamental Weyl chamber. This does however not impact on the general validity of the analysis below since the dimensions d_λ are invariant under the (shifted) action of W.

Using

$$d_{k\Lambda_1} = \binom{k+2n-1}{2n-1}, \tag{4.14}$$

we find

$$x = \gcd\Bigg\{ \binom{k+2n}{2n-1}, \binom{k+2n+1}{2n-1} + \binom{k+2n-1}{2n-1}, \\ \binom{k+2n+2}{2n-1} + \binom{k+2n-2}{2n-1}, \ldots, \binom{k+3n-1}{2n-1} + \binom{k+n+1}{2n-1} \Bigg\}. \tag{4.15}$$

Upon judicious use of the identity (4.6) this can be rewritten as

$$x = \gcd\left\{ \binom{k+n+1}{1}, \binom{k+n+2}{3}, \ldots, \binom{k+2n-1}{2n-3}, \binom{k+2n}{2n-1} \right\}. \tag{4.16}$$

Using a generalisation of eq. (4.7), which may be proven the same way, we arrive at eq. (4.11).

Alternatively, one may use a different set of generators of $\mathcal{I}_k$, corresponding to the boundary weights $\{k\Lambda_1 + \Lambda_i : i = 1, \ldots, n\}$. This leads to the same result, and we leave the details to the reader.

5. Exceptions

In section 3 we have argued that a necessary and sufficient condition for the dimension homomorphism (3.2) on the character ring $\mathbb{Z}[\chi_1, \ldots, \chi_n]$ to descend to $\mathcal{F}_k$ is that $\dim(\mathcal{I}_k) \subset x\mathbb{Z}$. Let us illustrate that this condition is sometimes too strong for a homomorphism $\mathcal{F}_k \to \mathbb{Z}/x\mathbb{Z}$ to exist, i.e. does not give the charge group as defined by eq. (3.1), by looking, for example, at the Lie algebra G_2 at level $k = 1$.

There are two integrable representations, corresponding to the weights Λ_0 (the unit of the fusion ring) and Λ_2. The nontrivial fusion rule reads

$$\phi_{\Lambda_2} \times \phi_{\Lambda_2} = \phi_{\Lambda_0} + \phi_{\Lambda_2}. \tag{5.1}$$

Substituting the appropriate dimensions we find that this equation is satisfied modulo $x = 41$. However, (3.4) would suggest $x = 1$, as the representations with boundary weights Λ_1 and $2\Lambda_2$ have dimensions 14 and 27, respectively.

To explain what is going on, consider the tensor product decomposition

$$\chi_{\Lambda_2}\chi_{\Lambda_2} = \chi_{\Lambda_0} + \chi_{\Lambda_2} + \chi_{\Lambda_1} + \chi_{2\Lambda_2}. \tag{5.2}$$

The truncation in going from the tensor product rule to eq. (5.1) therefore consists of setting $\chi_{\Lambda_1} + \chi_{2\Lambda_2}$ to zero rather than setting χ_{Λ_1} and $\chi_{2\Lambda_2}$ to zero separately, as we would have required in section 3.

The resolution of this apparent dilemma lies of course in the fact that the fundamental representation with weight Λ_1 is not integrable at level $k = 1$, so rather than writing the fusion ring as

$$\mathcal{F}_k = \frac{\mathbb{Z}[\chi_1, \chi_2]}{\mathcal{I}_k}, \tag{5.3}$$

$\mathfrak{g}$	k	x
B_n	1	$2\gcd\left(2n\left(n+1\right), n2^n, 2^{2n-1}-n-1\right)$
D_n	1	$\gcd\left(2n-1, 2^{2n-3}-n, 2^{2n-2}-1\right)$
E_6	1	26
E_7	1	3135
E_8	1	∞
E_8	2	4
F_4	1	649
G_2	1	41

Table 2: Exceptional cases

with $\mathcal{I}_k = \langle \chi_1, \chi_2^2 - \chi_1 - \chi_2 - 1 \rangle$ we can write

$$\mathcal{F}_k = \frac{\mathbb{Z}[\chi_2]}{\mathcal{I}'_k}\,, \tag{5.4}$$

with $\mathcal{I}'_k = \langle \chi_2^2 - \chi_2 - 1 \rangle$, and clearly $\dim(\mathcal{I}'_k) \subset 41\mathbb{Z}$. Thus this is the correct presentation to use for determining the charge group in this case.

Summarizing, when not all the fundamental representations are integrable at the level k one is considering, i.e. $k < \max\{a_i^\vee\}$, the presentation of the fusion ring as a quotient of the complete character ring is redundant, as it introduces artificial generators (the non-integrable fundamental representations) which must then be factored out again. In those cases it is a simple matter to compute x directly from (3.1), using the explicit fusion coefficients (which were calculated with the help of [32]). We have tabulated these values in table 2 whenever they disagree with the result of (3.5). Note that in many cases in which $k < \max\{a_i^\vee\}$, formula (3.5) continues to hold. [Note also that for E_8 at level 1 the charge group is just $\mathbb{Z}$ as there is only the trivial fusion rule.]

Finally, note that in almost all exceptional cases x is not a divisor of $k + \mathrm{h}^\vee$. For reasons to be discussed in section 7 these exceptional results therefore are not consistent with the prediction that would follow from a computation of the twisted K-theory.

6. Symmetry considerations

In this section we discuss some of the symmetry properties of the charges $D_\lambda = d_\lambda \mod x$, with x given by (3.5). To simplify the discussion let us extend the dimension formula for d_λ in (3.6) to all (not necessarily dominant) weights λ. It is obvious that d_λ (and hence D_λ) is invariant under both the Weyl group W and the group of outer automorphisms $\mathrm{Aut}^0(\mathfrak{g})$ of $\mathfrak{g}$. For elements $\Omega \in \mathrm{Aut}^0(\widehat{\mathfrak{g}})$, i.e. symmetries of the affine Dynkin diagram, we find however that

$$D_{\Omega\lambda} = e^{2\pi i (\Lambda_i|\rho)} D_\lambda\,, \tag{6.1}$$

where i is determined by $\widehat{\Lambda}_i = \Omega\widehat{\Lambda}_0$. We recall that

$$e^{2\pi i(\Lambda_i|\rho)} = \epsilon(w_\Omega)\,, \tag{6.2}$$

where $w_\Omega \in W$ is the Weyl group element determined by (see [30] for details)

$$\Omega\widehat{\lambda} = k(\Omega - 1)\widehat{\Lambda}_0 + w_\Omega\widehat{\lambda}\,. \tag{6.3}$$

This symmetry reflects a similar symmetry in the fusion rule coefficients

$$\begin{aligned} N_{\Omega(\lambda)\mu}{}^{\Omega(\nu)} &= N_{\lambda\mu}{}^{\nu}\,, \\ N_{\Omega(\lambda)\mu}{}^{\nu} &= N_{\lambda\Omega(\mu)}{}^{\nu}\,. \end{aligned} \tag{6.4}$$

It is well-known that $\mathrm{Aut}^0(\widehat{\mathfrak{g}})/\mathrm{Aut}^0(\mathfrak{g})$ is isomorphic to the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. The homomorphism for $\Omega \in \mathrm{Aut}^0(\widehat{\mathfrak{g}})$, such that $\Omega\widehat{\Lambda}_0 = \widehat{\Lambda}_i$, is explicitly given by

$$\Omega \mapsto z_i = e^{-2\pi i \Lambda_i \cdot H} \qquad \in Z(\mathfrak{g})\,. \tag{6.5}$$

Geometrically, acting with an element of the center corresponds to rotating the conjugacy class around which the D-brane is wrapped and hence should leave invariant the charge, up to a possible sign [16, 19]. In fact, in [16] (see also [19, 20] for SU(3), where this was referred to as the 'multiplet structure') it was shown that requiring the invariance (6.1), in the case of SU(N), leads to the same charge group. This is however not the case for the other Lie groups.

Finally, the charges D_λ have a symmetry under the affine Weyl group $W_{\mathrm{aff}} = T(Q^\vee) \rtimes W$, that reflects the symmetry in, e.g., the modular S-matrices and specialized characters (see (2.9))

$$D_{\hat{w}\cdot\lambda} = \epsilon(\hat{w}) D_\lambda\,, \qquad \hat{w} \in W_{\mathrm{aff}}\,. \tag{6.6}$$

The affine Weyl group W_{aff} and the outer automorphism group $\mathrm{Aut}^0(\widehat{\mathfrak{g}})$ combine into a group which projects onto the finite weight space as $T(Q^*) \rtimes W$, where Q^* is the lattice dual to Q. It turns out that, apart from a few low level exceptions, enforcing charge invariance (6.6) under $T(Q^*) \rtimes W$ determines the charge group in all cases. Including the expected invariance under affine outer automorphisms is therefore not in conflict with the charge group results.

Rather surprisingly, when studying the action of the extended affine Weyl group $W'_{\mathrm{aff}} = T(P) \rtimes W$ on D_λ, where P is the weight lattice of $\mathfrak{g}$ (in the simply laced case $P = Q^*$), one finds that (6.6) holds for $\hat{w} \in W'_{\mathrm{aff}}$ as well, in all cases except C_n, n not a power of 2, at certain levels.

In figure 1, we show as an example, the charges for C_2 at level 4 (for which (6.6) does hold for $\hat{w} \in W'_{\mathrm{aff}}$). The symmetry about the dashed line corresponds to the outer automorphism of $C_n^{(1)}$. It is easily seen to be an element of W of negative sign composed with a translation by $\left(k + \mathrm{h}^\vee\right)\Lambda_2 = 7\Lambda_2$ (note $\Lambda_2 \in Q^*$). The dotted lines correspond to more general symmetries. The lower line is an element of W of negative sign composed with a translation by $7\Lambda_1$ (but $\Lambda_1 \notin Q^*$). The upper dotted line corresponds to the composition of an element of W of negative sign composed with a translation by $7\,(-\Lambda_1 + \Lambda_2)$.

Requiring (6.6) to hold for all $\hat{w} \in W'_{\mathrm{aff}}$ for C_n would lead to the requirement that x is given by (3.5) with $y = \mathrm{lcm}\{1, 2, \ldots, 2n-1\} = \mathrm{lcm}\{1, 2, \ldots, \mathrm{h} - 1\}$, and thus would lead to a universal expression for all compact, simple, connected, simply-connected Lie groups

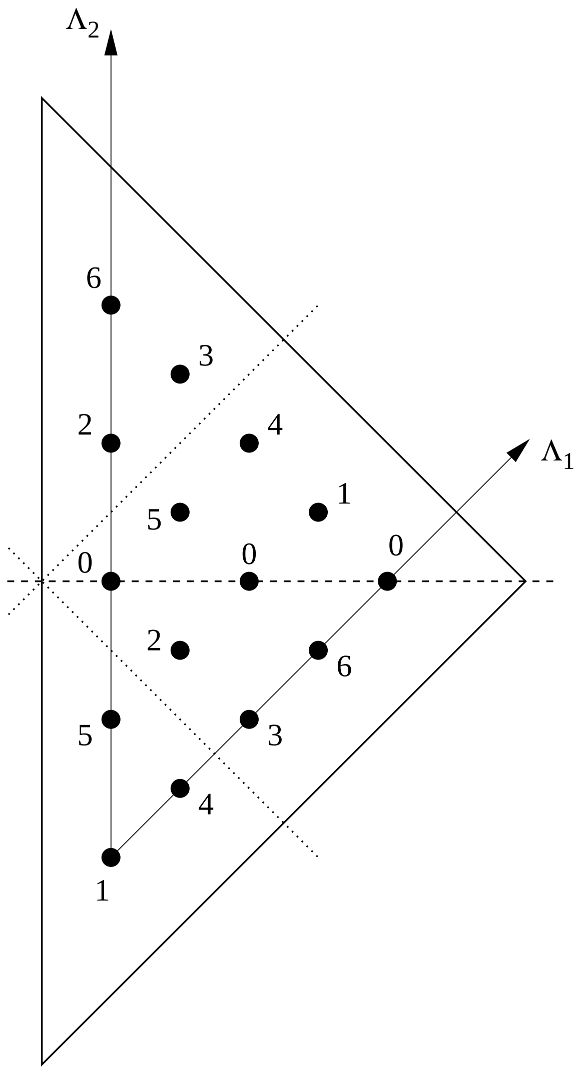

Figure 1: The integrable highest weights of C_2 at level 4 (shifted by ρ) and the charges of the corresponding symmetry preserving D-branes. The charges take values mod 7. Note that the charges are negated (mod 7) under reflection about the dashed and dotted lines.

(cf. the discussion in section 3). This is suggestive of the idea that maybe, in the case of C_n, the fusion rule constraints (3.1) on the charge group do not take into account all physical constraints that the D-brane charge group needs to satisfy. For this reason it would be very interesting to compute the twisted K-theory for C_n and make a comparison.

7. Conclusions and discussion

It has been argued [25]–[27] that the charges of D-branes on a manifold X in the background of a nontrivial NS-NS B-field, i.e. an element $[H] \in H^3(X,\mathbb{Z})$ (where $H = dB$ locally), are classified by twisted K-theory groups [36], at least in the large volume limit. In particular, for D-branes on a compact, simple, simply connected group manifold G, by $K^\bullet(G,(k+\mathrm{h}^\vee)[H_0])$, where $[H_0]$ is the generator of $H^3(G,\mathbb{Z}) \cong \mathbb{Z}$ and k is the level of the (supersymmetric) WZW model [14]. It has been verified that this proposal is consistent with the results based on boundary conformal field theory, i.e. the techniques used in this paper, for the the cases $G = \mathrm{SU}(2)$ [3, 14], $G = \mathrm{SU}(3)$ [14, 19] and $G = \mathrm{SU}(N)$ [14, 16].

In other cases, the comparison of the results of this paper with the corresponding twisted K-theory awaits a computation of the latter. In particular this would settle the issue of whether the consistency equations of [14], used in this paper, suffice to determine the charge group, in particular in the case of C_n (as discussed in section 6).

Beyond that, it would be very interesting to have a direct understanding of why the two approaches should lead to the same result. For instance, while it is obvious in twisted K-theory that x is a divisor of $k + \mathrm{h}^\vee$ (through the Atiyah-Hirzebruch spectral sequence [36]), there does not appear to be a simple explanation for this in our approach.

By a result of Freed, Hopkins and Telemann (cf. [37]) the fusion algebra $\mathcal{F}_k$ for the level-k WZW model can be identified with the twisted equivariant K-theory $K_G(G, (k+\mathrm{h}^\vee)[H_0])$. It would be very interesting to establish a direct link between the equivariant and non-equivariant twisted K-theory of G on the one hand, and the relation between the fusion algebra $\mathcal{F}_k$ and homomorphisms $\mathcal{F}_k \to \mathbb{Z}/x\mathbb{Z}$ on the other.

A particularly concrete presentation of twisted K-theory is as the Grothendieck K-theory of (isomorphism classes of) bundle gerbe modules [38]. Bundle gerbes show up naturally in the discussion of the WZW model [22, 39], and the bundle gerbe for $G = \mathrm{SU}(N)$ has recently been constructed in [22, 40]. It would be interesting to explicitly construct the bundle gerbe modules corresponding to a particular D-brane charge.

Finally, it would be interesting to extend the results of this paper to the non-symmetry preserving branes (i.e. twisted branes) [6, 19, 21], branes on coset spaces [15],[41]–[43], and beyond [44, 45]. We hope to report on this in the future.

A. Lie algebra notations

We summarize the Lie algebra notations used in this paper for convenience. Let $\mathfrak{g}$ denote a simple finite-dimensional Lie algebra of rank n, and $\widehat{\mathfrak{g}}$ its untwisted affine extension, i.e. $\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k$. The Cartan subalgebra of $\mathfrak{g}$ is denoted by $\mathfrak{h}$ and its dual by $\mathfrak{h}^*$. The set of positive roots in $\mathfrak{h}^*$ is denoted by Δ_+, the set of simple roots by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. The fundamental weights Λ_i of $\mathfrak{g}$ are defined by

$$(\Lambda_i | \alpha_j^\vee) = \delta_{ij} , \tag{A.1}$$

where $\alpha^\vee = 2\alpha/(\alpha|\alpha)$ is the co-root corresponding to α. Here we have normalized the nondegenerate positive bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ such that $(\theta|\theta) = 2$, where θ is the highest root of $\mathfrak{g}$.

The Weyl vector is denoted by ρ and is defined by

$$(\rho | \alpha_i^\vee) = 1 , \tag{A.2}$$

for all $i = 1, \ldots, n$, i.e., $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = \sum_{i=1}^{n}\Lambda_i$. The marks $\{a_i\}_{i=1}^{n}$ and co-marks $\{a_i^\vee\}_{i=1}^{n}$ are given by

$$\theta = \sum_{i=1}^{n} a_i \alpha_i = \sum_{i=1}^{n} a_i^\vee \alpha_i^\vee \,. \tag{A.3}$$

We denote the Coxeter and dual Coxeter numbers by h and $\mathrm{h}^\vee$, respectively (see table 1), the root and co-root lattices by Q and $Q^\vee$, and the lattices of integral weights and dominant integral weights by P and P_+, and integrable weights at level k by $P_+^{(k)} = \{\lambda \in P_+ : (\lambda|\theta) \leqslant k\}$.

For the affine Lie algebra $\widehat{\mathfrak{g}}$ we use the same notations except that we put a hat on the symbols. To simplify notations, however, we will sometimes simply use the notation λ for an integrable weight $\hat{\lambda} = (\lambda, k)$, when the level k is understood.

We denote the Weyl groups of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ by W and $\widehat{W}$, respectively, and the shifted Weyl group action by a dot, i.e.

$$w \cdot \lambda = w(\lambda + \rho) - \rho \,, \tag{A.4}$$

and similarly for $\hat{w} \in \widehat{W}$. The determinant of a Weyl group element w is denoted by $\epsilon(w)$. The projection of $\widehat{W}$ onto $\mathfrak{h}^*$ is given by

$$W_{\mathrm{aff}} = T(Q^\vee) \rtimes W \,, \tag{A.5}$$

where $T(\Gamma)$ denotes the translation group of the lattice Γ, and the semi-direct product structure is given by the relation $w t_\alpha w^{-1} = t_{w\alpha}$, i.e. $(t_\gamma, w) \circ (t_{\gamma'}, w') = (t_\gamma t_{w\gamma'}, ww')$. In particular, $t_{\theta^\vee} = r_0 r_\theta$. Explicitly, the shifted action of $t_\gamma \in W_{\mathrm{aff}}\,,\ \gamma \in Q^\vee$, is given by

$$t_\gamma \cdot \lambda = \lambda + (k + \mathrm{h}^\vee)\gamma \,. \tag{A.6}$$

B. Generating subsets

In this appendix we provide a subset of the boundary weights, i.e. weights satisfying $(\lambda|\theta) = k+1$, which we believe form a generating set for the fusions ideals $\mathcal{I}_k$. For A_n and C_n this has been proven, in the other cases we tested numerically for $k < 5000$ that these sets give rise to the result (3.5).

When k is sufficiently small, some of the weights given below will not lie in the fundamental chamber — these weights should be discarded to get a generating set, except in the following cases where it is additionally required to include another weight. This occurs with E_7 at levels 2 and 3, adding $(0,0,0,1,0,0,0)$ and $(0,0,0,0,0,0,2)$ respectively, and E_8 at levels 4 and 12, adding respectively $(0,1,0,0,0,0,1,0)$ and $(0,2,1,0,0,0,0,1)$. However, it may be verified that for E_8 at level 2, there is no set of weights in the fundamental chamber whose dimensions have the greatest common divisor given in (3.5). In this case it is necessary to include weights outside the fundamental chamber (extending the dimension formula in the obvious way).

Fundamental weights are denoted by Λ_i. For the exceptional algebras, we have listed the weights appearing in the generating sets in terms of their Dynkin labels, for clarity, using the conventions of [30].

$\boldsymbol{A_n}$

$$\{k\Lambda_1 + \Lambda_i : i = 1, \ldots, n\} \ .$$

$\boldsymbol{B_n}$

$$\begin{aligned}&\{(k+1)\,\Lambda_1, (k-1)\,\Lambda_1 + 2\Lambda_n\}\\ &\qquad \cup \{(k-1)\,\Lambda_1 + \Lambda_i, (k-2)\,\Lambda_1 + \Lambda_i + \Lambda_n : i = 2, \ldots, n-1\} \ .\end{aligned}$$

$\boldsymbol{C_n}$

$$\{k\Lambda_1 + \Lambda_i : i = 1, \ldots, n\} \ .$$

$\boldsymbol{D_n}$

$$\begin{aligned}&\{k\Lambda_1 + \Lambda_n, (k-1)\,\Lambda_1 + 2\Lambda_n, (k-1)\,\Lambda_1 + \Lambda_{n-1} + \Lambda_n, (k-3)\,\Lambda_1 + \Lambda_{n-1} + 3\Lambda_n\}\\ &\quad \cup \{(k-1)\,\Lambda_1 + \Lambda_i, (k-3)\,\Lambda_1 + \Lambda_i + 2\Lambda_n : i = 2, \ldots, n-2\}\\ &\quad \cup \{(k-3)\,\Lambda_1 + \Lambda_i + \Lambda_j : 2 \leqslant i < j \leqslant n-2\} \ .\end{aligned}$$

$\boldsymbol{E_6}$ For k odd,

$$\begin{aligned}\Bigg\{&\left(0,0,0,0,0,\frac{k+1}{2}\right), \left(0,0,0,0,2,\frac{k-1}{2}\right), \left(1,0,0,0,1,\frac{k-1}{2}\right), \left(0,1,0,1,0,\frac{k-3}{2}\right),\\ &\left(0,0,0,0,6,\frac{k-5}{2}\right), \left(0,0,0,3,0,\frac{k-5}{2}\right), \left(3,0,0,0,3,\frac{k-5}{2}\right), \left(1,0,1,0,4,\frac{k-7}{2}\right)\Bigg\}.\end{aligned}$$

For k even,

$$\begin{aligned}\Bigg\{&\left(0,0,0,0,1,\frac{k}{2}\right), \left(0,0,0,0,3,\frac{k-2}{2}\right), \left(0,0,0,1,1,\frac{k-2}{2}\right),\\ &\left(0,0,1,0,0,\frac{k-2}{2}\right), \left(0,2,0,0,1,\frac{k-4}{2}\right), \left(1,0,1,0,1,\frac{k-4}{2}\right),\\ &\left(0,1,0,0,5,\frac{k-6}{2}\right), \left(0,1,0,2,1,\frac{k-6}{2}\right), \left(2,0,0,1,3,\frac{k-6}{2}\right)\Bigg\}.\end{aligned}$$

$\boldsymbol{E_7}$ For k odd,

$$\begin{aligned}\Bigg\{&\left(\frac{k+1}{2},0,0,0,0,0,0\right), \left(\frac{k-1}{2},0,0,0,1,0,0\right),\\ &\left(\frac{k-3}{2},0,1,0,0,0,0\right), \left(\frac{k-5}{2},0,0,2,0,0,0\right), \left(\frac{k-7}{2},0,0,0,0,8,0\right),\\ &\left(\frac{k-7}{2},0,0,0,4,0,0\right), \left(\frac{k-7}{2},0,1,0,2,0,0\right), \left(\frac{k-9}{2},0,0,2,0,4,0\right),\\ &\left(\frac{k-9}{2},0,1,0,0,6,0\right), \left(\frac{k-9}{2},0,1,0,2,2,0\right)\Bigg\}.\end{aligned}$$

For k even,

$$\left\{\left(\frac{k}{2},0,0,0,0,1,0\right),\left(\frac{k-2}{2},1,0,0,0,0,0\right),\right.$$
$$\left(\frac{k-2}{2},0,0,0,0,1,1\right),\left(\frac{k-4}{2},1,0,0,0,2,0\right),\left(\frac{k-4}{2},1,0,0,1,0,0\right),$$
$$\left(\frac{k-4}{2},0,0,1,0,0,1\right),\left(\frac{k-6}{2},1,0,1,0,1,0\right),\left(\frac{k-6}{2},0,0,1,1,0,1\right),$$
$$\left(\frac{k-8}{2},1,0,0,3,0,0\right),\left(\frac{k-8}{2},0,0,0,0,7,1\right),\left(\frac{k-8}{2},0,0,0,3,1,1\right),$$
$$\left.\left(\frac{k-10}{2},1,0,1,0,5,0\right),\left(\frac{k-10}{2},1,0,1,1,3,0\right)\right\}.$$

$\boldsymbol{E_8}$ For k odd,

$$\left\{\left(\frac{k+1}{2},0,0,0,0,0,0,0\right),\left(\frac{k-1}{2},0,0,0,0,0,1,0\right),\left(\frac{k-3}{2},0,1,0,0,0,0,0\right),\right.$$
$$\left(\frac{k-5}{2},0,0,0,1,0,0,0\right),\left(\frac{k-5}{2},0,1,0,0,0,1,0\right),\left(\frac{k-5}{2},0,0,0,0,2,0,0\right),$$
$$\left(\frac{k-7}{2},0,0,0,0,2,0,0\right),\left(\frac{k-7}{2},0,1,0,0,0,2,0\right),\left(\frac{k-9}{2},0,0,0,0,1,0,2\right),$$
$$\left(\frac{k-9}{2},0,1,0,1,0,0,0\right),\left(\frac{k-9}{2},1,0,0,0,1,0,1\right),\left(\frac{k-11}{2},0,0,0,0,0,6,0\right),$$
$$\left(\frac{k-11}{2},0,0,0,2,0,0,0\right),\left(\frac{k-13}{2},0,0,0,1,0,4,0\right),\left(\frac{k-15}{2},0,0,0,0,4,0,0\right),$$
$$\left(\frac{k-15}{2},0,0,2,0,0,3,0\right),\left(\frac{k-17}{2},0,0,0,0,0,0,6\right),\left(\frac{k-17}{2},0,0,0,3,0,0,0\right),$$
$$\left.\left(\frac{k-17}{2},0,1,0,1,2,0,0\right),\left(\frac{k-19}{2},0,1,2,0,0,0,2\right),\left(\frac{k-19}{2},0,2,0,2,0,0,0\right)\right\}.$$

For k even,

$$\left\{\left(\frac{k-2}{2},0,0,0,0,0,0,1\right),\left(\frac{k-2}{2},1,0,0,0,0,0,0\right),\left(\frac{k-4}{2},0,0,1,0,0,0,0\right),\right.$$
$$\left(\frac{k-6}{2},0,0,0,0,1,0,1\right),\left(\frac{k-6}{2},1,0,0,0,0,2,0\right),\left(\frac{k-8}{2},0,0,0,0,0,0,3\right),$$
$$\left(\frac{k-8}{2},0,0,0,1,0,0,1\right),\left(\frac{k-8}{2},0,0,1,0,0,2,0\right),\left(\frac{k-8}{2},3,0,0,0,0,0,0\right),$$
$$\left(\frac{k-10}{2},0,1,0,0,1,0,1\right),\left(\frac{k-10}{2},0,0,0,1,0,1,1\right),\left(\frac{k-10}{2},1,0,1,0,0,0,1\right),$$
$$\left(\frac{k-12}{2},0,0,0,0,0,5,1\right),\left(\frac{k-12}{2},0,0,1,1,0,1,0\right),\left(\frac{k-14}{2},0,0,1,0,1,3,0\right),$$
$$\left(\frac{k-16}{2},0,0,1,0,3,0,0\right),\left(\frac{k-16}{2},0,1,1,0,1,2,0\right),\left(\frac{k-18}{2},0,0,2,0,0,0,3\right),$$
$$\left.\left(\frac{k-18}{2},0,1,0,0,0,0,5\right),\left(\frac{k-18}{2},0,1,0,2,0,0,1\right),\left(\frac{k-18}{2},0,2,0,0,2,0,1\right)\right\}.$$

$\boldsymbol{F_4}$ For k odd,

$$\left\{\left(\frac{k+1}{2},0,0,0\right),\left(\frac{k-1}{2},0,0,2\right),\left(\frac{k-1}{2},0,1,0\right),\left(\frac{k-3}{2},1,0,1\right),\right.$$
$$\left.\left(\frac{k-3}{2},0,2,0\right),\left(\frac{k-5}{2},0,1,4\right),\left(\frac{k-5}{2},1,0,3\right)\right\}.$$

For k even,

$$\left\{\left(\frac{k}{2},0,0,1\right),\left(\frac{k-2}{2},0,1,1\right),\left(\frac{k-2}{2},1,0,0\right),\left(\frac{k-4}{2},0,0,5\right),\right.$$
$$\left.\left(\frac{k-4}{2},1,0,2\right),\left(\frac{k-4}{2},0,2,1\right),\left(\frac{k-4}{2},1,1,0\right),\left(\frac{k-6}{2},1,0,4\right)\right\}.$$

$\boldsymbol{G_2}$ For k odd,

$$\left\{\left(\frac{k+1}{2},0\right),\left(\frac{k-1}{2},2\right),\left(\frac{k-3}{2},4\right)\right\}.$$

For k even,

$$\left\{\left(\frac{k}{2},1\right),\left(\frac{k-2}{2},3\right),\left(\frac{k-4}{2},5\right)\right\}.$$

C. Proof using fusion potentials

It is well-known that in the case of fusion rings, the affine algebraic variety defined by the fusion ideal is a so-called jacobian variety, i.e., if $\mathcal{F}=\mathbb{C}[x_1,\ldots,x_N]/\mathcal{I}$ we can find a so-called fusion potential $V(x_1,\ldots,x_N)$ such that $\mathcal{I}=\langle P_1,\ldots,P_N\rangle$, where $P_i\equiv\partial V/\partial x_i$ [29, 33, 34, 35, 46]. Since the fusion potential provides a set of generators of the fusion ideal, it can be used to give an alternative proof of (3.5). While an expression for the fusion potential is known, in principle, for all compact, simple, connected, simply-connected Lie groups G [46], we have not been able to make effective use of it for cases other than A_n and C_n. In this appendix we present an alternative proof for those two cases.

The fusion potential for A_n takes the form [29]

$$V(\chi_1,\ldots,\chi_n)=\frac{1}{k+n+1}\sum_{i=1}^{n+1}q_i^{k+n+1}\,, \tag{C.1}$$

where the (formal) variables q_i are related to the standard (overcomplete) basis $\{\epsilon_i\}_{i=1}^{n+1}$ for the weight space of A_n by $q_i=\exp(\epsilon_i)$, with the constraint $q_1\cdots q_{n+1}=1$. In terms of these variables the characters χ_m of the fundamental representations $L(\Lambda_m)$, $(m=1,\ldots,n)$, are given in terms of the elementary symmetric polynomials, i.e.

$$\chi_m=\sum_{1\leqslant i_1<\ldots<i_m\leqslant n+1}q_{i_1}\cdots q_{i_m}\,. \tag{C.2}$$

Defining

$$V_m=\frac{1}{m}\sum_{i=1}^{n+1}q_i^m\,, \tag{C.3}$$

we can form the generating function

$$V(t) = \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m = \log\left(\sum_{i=0}^{n+1} \chi_i t^i\right) , \tag{C.4}$$

where we have set $\chi_0 = 1 = \chi_{n+1}$ for convenience. The fusion potential is now (up to a sign) the coefficient of t^{k+n+1} in $V(t)$, and the generators of the fusion ideal are the derivatives of this coefficient with respect to the χ_i ($i = 1, \ldots, n$). Differentiating $V(t)$ with respect to χ_j, and evaluating at $q_i = 1$ (i.e. at the point where $\chi_i = d_{\Lambda_i} = \binom{n+1}{i}$), gives

$$\left.\frac{\partial V(t)}{\partial \chi_j}\right|_{\chi_i = d_{\Lambda_i}} = \frac{t^j}{(1+t)^{n+1}} . \tag{C.5}$$

Taylor expanding this around 0 and extracting the coefficient of t^{k+n+1} gives the values of the generators of the fusion ideal under the dimension map, and hence determines x. The result is

$$x = \gcd\left\{\binom{k+2n+1-j}{n} : j = 1, \ldots, n\right\} = \gcd\left\{\binom{k+n+j}{n} : j = 1, \ldots, n\right\} , \tag{C.6}$$

in accordance with eq. (4.10).

The proof above for A_n easily generalizes to C_n. The fusion potential is given by [34, 35]

$$V(\chi_1, \chi_2, \ldots, \chi_n) = \frac{1}{k+n+1} \sum_{i=1}^{n} \left(q_i^{k+n+1} + q_i^{-(k+n+1)}\right) , \tag{C.7}$$

where, in terms of an orthonormal basis $\{\epsilon_i\}_{i=1}^n$ for the weight space of C_n, the variables q_i are given by $q_i = \exp(\epsilon_i)$, and the characters χ_j ($j = 1, \ldots, n$) are related to the variables q_i by

$$\chi_j = E_j - E_{j-2} \qquad (E_j = 0 \text{ when } j < 0), \tag{C.8}$$

and E_j is defined by

$$\sum_{j=0}^{\infty} E_j t^j = \prod_{i=1}^{n} (1 + q_i t)\left(1 + q_i^{-1} t\right) . \tag{C.9}$$

In particular we have $E_j = E_{2n-j}$, and $E_j = 0$ for $j > 2n$. Another consequence of this is that $\chi_j + \chi_{2n+2-j} = 0$, so there are indeed only n linearly independent characters (in agreement with the arguments of the fusion potential V).

In analogy to A_n, we define $V_m = \frac{1}{m}\sum_{i=1}^n \left(q_i^m + q_i^{-m}\right)$, and form the generating function

$$V(t) = \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m = \log\left(\sum_{j=0}^{2n} E_j t^j\right) . \tag{C.10}$$

Differentiating with respect to the χ_i gives

$$\frac{\partial V(t)}{\partial \chi_i} = \frac{t^i + t^{i+2} + \cdots + t^{2n-2-i} + t^{2n-i}}{\sum_{j=0}^{2n} E_j t^j} . \tag{C.11}$$

Evaluating this at $q_i = 1$, i.e. at the point $\chi_i = d_{\Lambda_i} = \binom{2n}{i} - \binom{2n}{i-2}$ (thus $E_i = \binom{2n}{i}$), we find

$$\left.\frac{\partial V\,(t)}{\partial \chi_i}\right|_{\chi_j = d_{\Lambda_j}} = \frac{t^i + t^{i+2} + \cdots + t^{2n-2-i} + t^{2n-i}}{(1+t)^{2n}}\,. \tag{C.12}$$

Taylor expanding about 0, and extracting the coefficient of t^{k+n+1}, gives the charge group parameter as

$$\begin{aligned}
x &= \gcd\left\{ \sum_{j=0}^{n-i} \binom{k+3n-i-2j}{2n-1} : i = 1, \ldots, n \right\} \\
&= \gcd\left\{ \sum_{j=0}^{i-1} \binom{k+2n+1-i+2j}{2n-1} : i = 1, \ldots, n \right\} \\
&= \gcd\bigg\{ \binom{k+2n}{2n-1}, \binom{k+2n+1}{2n-1} + \binom{k+2n-1}{2n-1}, \\
&\qquad \binom{k+2n+2}{2n-1} + \binom{k+2n-2}{2n-1}, \ldots, \binom{k+3n-1}{2n-1} + \binom{k+n+1}{2n-1} \bigg\},
\end{aligned}$$

as in eq. (4.15).

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